Math Virtual Learning

Precalculus with Trigonometry

May 15, 2020

Precalculus with Trigonometry
Lesson: May 15th, 2020

Objective/Learning Target:
Students will be able to find powers and roots of complex numbers written in trigonometric form.

Let's Get Started:

Before watching the video, recall from the previous lesson how to determine the r and θ values of a complex number given in standard form.

$$r = \sqrt{a^2 + b^2}, \text{ and } \tan\theta = b/a.$$

Watch Video: Complex Number to a Power Using DeMoivre's Theorem

So in essence, DeMoivre's Theorem is a much more efficient method to determine the value of a power of a complex number.

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$
$$= r^n(\cos n\theta + i\sin n\theta).$$

Example #1: Use DeMoivre's Theorem to find $(-1+\sqrt{3}i)^{12}$.

Solution

First convert the complex number to trigonometric form using

$$r=\sqrt{(-1)^2+(\sqrt{3})^2}=2 \quad \text{and} \quad \theta=\arctan\frac{\sqrt{3}}{-1}=\frac{2\pi}{3}.$$

So, the trigonometric form is

$$z=-1+\sqrt{3}i=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right).$$

Then, by DeMoivre's Theorem, you have

$$\begin{aligned}(-1+\sqrt{3}i)^{12}&=\left[2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\right]^{12}\\&=2^{12}\left[\cos\frac{12(2\pi)}{3}+i\sin\frac{12(2\pi)}{3}\right]\\&=4096(\cos 8\pi+i\sin 8\pi)\\&=4096(1+0)\\&=4096.\end{aligned}$$

Question: Since there is a method to find a power of a complex number written in trigonometric form, is there a method to determine roots?

Answer: Absolutely

Watch the following video to see how to determine a root using the trigonometric form of a complex number.

Video: Finding nth Roots of a Complex Number

Definition of an nth Root of a Complex Number

The complex number $u = a + bi$ is an **nth root** of the complex number z if

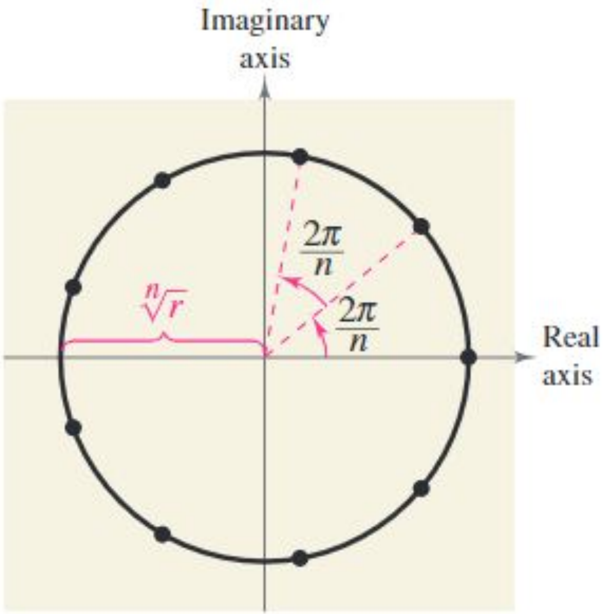
$$z = u^n = (a + bi)^n.$$

Finding nth Roots of a Complex Number

For a positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots given by

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$.



Example #2: Find all sixth roots of 1.

Solution

First write 1 in the trigonometric form $1 = 1(\cos 0 + i \sin 0)$. Then, by the nth root formula, with $n = 6$ and $r = 1$, the roots have the form

$$\sqrt[6]{1}\left(\cos\frac{0 + 2\pi k}{6} + i\sin\frac{0 + 2\pi k}{6}\right) = \cos\frac{\pi k}{3} + i\sin\frac{\pi k}{3}.$$

So, for $k = 0, 1, 2, 3, 4,$ and 5, the sixth roots are as follows. (See Figure 6.49.)

$$\cos 0 + i \sin 0 = 1$$

$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Increment by $\frac{2\pi}{n} = \frac{2\pi}{6} = \frac{\pi}{3}$

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\cos \pi + i \sin \pi = -1$$

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

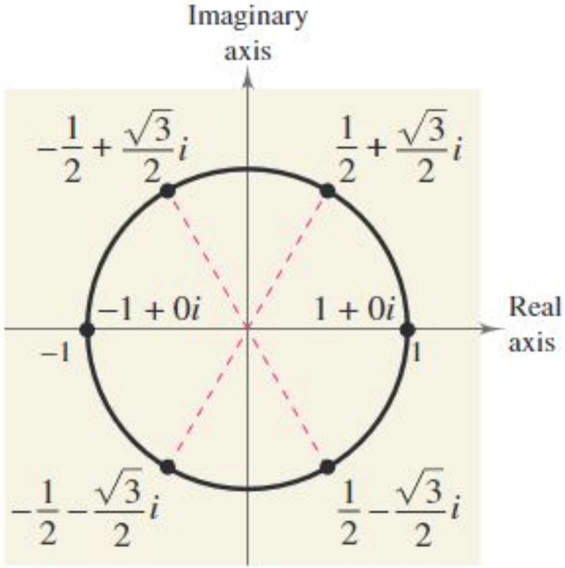


FIGURE 6.49

Example #3: Find the three cube roots of $z = -2 + 2i$.

Solution

Because z lies in Quadrant II, the trigonometric form of z is

$$z = -2 + 2i$$

$$= \sqrt{8}\,(\cos 135° + i \sin 135°). \qquad \theta = \arctan\left(\frac{2}{-2}\right) = 135°$$

By the formula for nth roots, the cube roots have the form

$$\sqrt[6]{8}\left(\cos\frac{135° + 360°k}{3} + i \sin\frac{135° + 360°k}{3}\right).$$

Finally, for $k = 0$, 1, and 2, you obtain the roots

$$\sqrt[6]{8}\left(\cos\frac{135° + 360°(0)}{3} + i \sin\frac{135° + 360°(0)}{3}\right) = \sqrt{2}(\cos 45° + i \sin 45°)$$

$$= 1 + i$$

$$\sqrt[6]{8}\left(\cos\frac{135° + 360°(1)}{3} + i \sin\frac{135° + 360°(1)}{3}\right) = \sqrt{2}(\cos 165° + i \sin 165°)$$

$$\approx -1.3660 + 0.3660i$$

$$\sqrt[6]{8}\left(\cos\frac{135° + 360°(2)}{3} + i \sin\frac{135° + 360°(2)}{3}\right) = \sqrt{2}(\cos 285° + i \sin 285°)$$

$$\approx 0.3660 - 1.3660i.$$

See Figure 6.50.

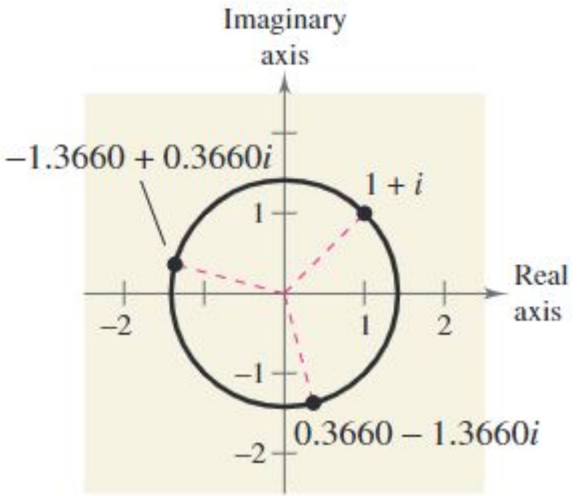


FIGURE 6.50

Practice

On a separate piece of paper, do each of the following problems. Answers will be provided on the next page.

Use Demoivre's Theorem to find the indicated power.

1. $(1 + i)^5$

2. $(3 - 2i)^8$

Use the formula on a previous screen to find the indicated roots of a complex number and write each root in standard form.

3. Square roots of $16(\cos 60° + i \sin 60°)$

4. Fifth roots of $32\left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right)$

Practice - **ANSWERS**

1. $-4-4i$

2. $-239+28560i$

3. $4(\cos 30^\circ + i\sin 30^\circ) = 2\sqrt{3}+2i$

   $4(\cos 210^\circ + i\sin 210^\circ) = -2\sqrt{3}-2i$

4. $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{3}+i$

   $2\left(\cos\frac{17\pi}{30} + i\sin\frac{17\pi}{30}\right) = -0.4158+1.9563i$

   $2\left(\cos\frac{29\pi}{30} + i\sin\frac{29\pi}{30}\right) = -1.9890+0.2091i$

   $2\left(\cos\frac{41\pi}{30} + i\sin\frac{41\pi}{30}\right) = -0.8135-1.8271i$

   $2\left(\cos\frac{53\pi}{30} + i\sin\frac{53\pi}{30}\right) = 1.4863-1.3383i$

Additional Resource Videos:

DeMoivre's Theorem: Raising a Complex Number to a Power, Ex 1

DeMoivre's Theorem: Raising a Complex Number to a Power, Ex 2

Roots of Complex Numbers, Ex 1

Additional Practice:

Complex Numbers Rectangular and Trig Form

Problems 19-24

Please note that when it asks for Polar Form, it means Trig Form

DeMoivre's Theorm Worksheet